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Generalized Invariants and Quantum Evolution of Open Fermionic System

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Abstract

Open systems acquire time-dependent coupling constants through interaction with an external field or environment. We generalize the Lewis-Riesenfeld invariant theorem to open system of quantum fields after second quantization. The generalized invariants and thereby the quantum evolution are found explicitly for time-dependent quadratic fermionic systems. The pair production of fermions is computed and other physical implications are discussed.

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The dynamical evolution of the Universe in time has to be examined carefully to understand the phase transitions that are crucial to the cosmological scenarios. The phase transition may be affecting topologically unstable structures in the Universe and these may have important consequences for the inflationary Universe. The observational aspects like baryon asymmetry may require the presence of time-dependent gauge fields. The time evolution of the Universe would require that both scalar and fermionic fields are time dependent through their interactions with the gravity. Furthermore one may need to calculate the pair production of fermions due to the oscillating inflatons. This may provide a possible mechanism for preheating mechanism for fermionic systems.

All these time-dependent processes in the Universe would be happening when the system is not in equilibrium and there is no fixed temperature. We have a system that is time-dependent and in a non-equilibrium state. Similar time-dependent and rapidly varying processes in the laboratory, that simulate the conditions at the early evolutionary stage of the Universe, relate to the heavy ion reactions at the relativistic heavy ion facility at Brookhaven. It is intended that in colliding heavy ion (U on U) reactions there will be a deconfining phase transition from nucleonic to a quark-gluon system that exists for a very brief period of time before making a phase transition back to the baryonic matter. The time evolution of the formation and dissolution of the quark-gluon plasma is a highly non-equilibrium process that requires a careful consideration of the time dependence of the coupling of both the fermionic and bosonic systems.

The coupling constants or parameters of a system depend on time implicitly through the interaction when it interacts with an environment. Any system whose coupling constants or parameters depend on time explicitly can be regarded as a subsystem of a larger closed system and belongs to an open system. To describe properly the quantum states of such open system one may find the evolution of the system plus environment and integrate out the degrees of freedom corresponding to the environment. Or one may rely on field theoretical methods such as closed time path integral [1] or thermo field dynamics [2].

Even though the thermo field dynamics is a real time operator formalism it has not been applied to a real problem to investigate the time evolution. Hence we choose instead to use the functional Schrödinger picture approach [3]. It is conceivable that this may provide some help in using the thermo field dynamics. Even the quantum evolution of a time-dependent system has also been an outstanding problem in quantum mechanics. A frequently used method is to find trial wave functions (a variational approach) with undetermined time-dependent coefficients and to fix these coefficients by solving the Schrödinger equation. Lewis and Riesenfeld introduced a systematic method to find the operators that satisfy the quantum Liouville-von Neumann equation and to solve for the exact quantum states as their eigenstates [4]. This method has been applied successfully to some of the time-dependent systems, in particular, the harmonic oscillator systems.

The Lewis-Riesenfeld invariant theorem has been applied successfully [5] to study the quantum evolution of scalar fields in an expanding Friedman-Robertson-Walker universe. A variety of results were obtained including the vacuum state and the Wightman functions with respect to the vacuum, thermal equilibrium and coherent states. However, as suggested above the overall evolution of the universe would require not only the time dependence of the scalar field but also that of a fermion field. Since the fermion field is a four component spinor, it appears natural to use the Lewis-Riesenfeld invariant theorem to the fermion field taking proper account of the fact that the spinor components lead to a set of coupled equations. The usual procedure of writing the spinor field in terms of creation and annihilation operators is adopted. The results of quantum evolution of the fermion field are presented. We give a brief overview of the application to the scalar field and then a detailed account of the method as applied to the fermion system is given.

The open system of the quantum field is described by the functional Schrödinger equation [3]

$$i\frac{\partial}{\partial t}|\Psi(\phi,t)\rangle = \hat{H}(\phi, -i\frac{\delta}{\delta\phi}, t)|\Psi(\phi,t)\rangle, \tag{1}$$

where ϕ represents a scalar or fermion field. Our stratagem is to express the Hamiltonian in the second quantization as a sum of infinite number of finite dimensional quantum systems, which are decoupled or coupled depending on the type of interactions. And then we apply the Lewis-Riesenfeld theorem to each finite dimensional system and find the generalized invariants and thereby the evolution of the second quantized Hamiltonian.

According to the Lewis-Riesenfeld invariant theorem, the invariant operator satisfying the Liouville-von Neumann equation for a time-dependent Hamiltonian $\hat{H}(t)$

$$i\frac{d}{dt}\hat{\mathcal{O}}(t) = i\frac{\partial}{\partial t}\hat{\mathcal{O}}(t) + [\hat{\mathcal{O}}(t), \hat{H}(t)] = 0, \tag{2}$$

has time-dependent eigenstates and time-independent eigenvalues

$$\hat{\mathcal{O}}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle. \tag{3}$$

And the exact quantum evolution of the system

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle, \tag{4}$$

is given by a superposition of states $|\lambda_n, t\rangle$, up to time-dependent phase factors.

The theorem was originally proved by Lewis and Riesenfeld and applied to a time-dependent harmonic oscillator [4]. We now give a simple proof of the theorem. One can apply the operator eq. (2) to the eigenstates $|\lambda_n, t\rangle$ given by eq. (3) and first show that $\frac{\partial}{\partial t}\lambda_n = 0$, and then obtain the following identity

$$\Big(\hat{\mathcal{O}}(t) - \lambda_n \hat{I}\Big)\Big[\Big(i\frac{\partial}{\partial t} - \hat{H}(t)\Big)|\lambda_n, t\rangle\Big] = 0. \tag{5}$$

Here, we use the eigenvalue equation (3) and $\hat{I}$ denotes the identity operator. This implies that the quantity in the square bracket should belong to the kernal of the operator in the parenthesis, that is $|\lambda_n, t\rangle$:

$$\Big(i\frac{\partial}{\partial t} - \hat{H}(t)\Big)|\lambda_n, t\rangle = \alpha_n(t)|\lambda_n, t\rangle. \tag{6}$$

Then it follows readily that

$$\alpha_n(t) = \langle\lambda_n, t|\Big(i\frac{\partial}{\partial t} - \hat{H}(t)\Big)|\lambda_n, t\rangle, \tag{7}$$

and the quantum state

$$|\Psi_n, t\rangle = e^{i\int \alpha_n(t)}|\lambda_n, t\rangle, \tag{8}$$

is an exact solution of eq. (4). The most general state is a superposition of states like the ones given by eq. (8).

It should be pointed out that the Lewis-Riesenfeld theorem can also be applied to infinite-dimensional quantum systems such as the quantum fields when they are expressed as a sum of finite dimensional systems. The second quantized Hamiltonian of quantum fields has such a form. Another important observation is that the theorem can also be applied to bosonic or fermionic systems when the commutators or anticommutators are appropriately used for boson and fermion operators, respectively. We shall consider a scalar field with a time-dependent mass and a time-dependent fermion system with the most general quadratic form as open systems.

A massive scalar field with a time-dependent mass described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2(t)}{2}\phi^2 \tag{9}$$

has the second quantized Hamiltonian in the standard basis of the Minkowski spacetime

$$\hat{H}(t) = \sum_\alpha \Omega_\alpha^{(D)}(t)\Big(\hat{a}_\alpha^\dagger\hat{a}_\alpha + \frac{1}{2}\Big) + \Omega_\alpha(t)\Big(\hat{a}_\alpha^{\dagger 2} + \hat{a}_\alpha^2\Big) \equiv \sum_\alpha \hat{H}_\alpha(t). \tag{10}$$

Here, α denotes collectively the Fourier sine/cosine modes, and

$$\Omega_\alpha^{(D)}(t) = \frac{\omega_\alpha^2}{2} + \frac{1}{2}, \quad \Omega_\alpha(t) = \frac{\omega_\alpha^2}{4} - \frac{1}{4}, \tag{11}$$

where

$$\omega_\alpha^2 = m^2(t) + \mathbf{k}^2. \tag{12}$$

The creation and annihilation operators for each mode satisfy the usual commutation relation

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha,\beta}. \tag{13}$$

The Lewis-Riesenfeld theorem applied to $\hat{H}_\alpha(t)$ gives rise to a pair of invariants [5]

$$\hat{A}_\alpha(t) = f_\alpha^{(-)}(t)\hat{a}_\alpha + f_\alpha^{(+)}(t)\hat{a}_\alpha^\dagger, \quad \hat{A}_\alpha^\dagger(t) = \text{h.c.}, \tag{14}$$

where

$$f_\alpha^{(-)}(t) = \frac{1}{\sqrt{2}}(-i\dot{\varphi}_\alpha^*(t) + \varphi_\alpha^*(t)), \quad f_\alpha^{(+)}(t) = \frac{1}{\sqrt{2}}(-i\dot{\varphi}_\alpha^*(t) - \varphi_\alpha^*(t)), \tag{15}$$

$$\ddot{\varphi}_\alpha(t) + \omega_\alpha^2(t)\varphi_\alpha(t) = 0. \tag{16}$$

It should be remarked that $\varphi_{\mathbf{k}}(t)$ satisfies the classical equation for the $\mathbf{k}$-mode. The equal time commutator

$$[\hat{A}_\alpha(t), \hat{A}^\dagger_\beta(t)] = \delta_{\alpha,\beta}, \tag{17}$$

is satisfied for all time for a complex solution due to the wronskian of eq. (16):

$$i\Big(\dot{\varphi}_\alpha(t)\varphi^*_\alpha(t) - \dot{\varphi}^*_\alpha\varphi_\alpha(t)\Big) = 1. \tag{18}$$

The number operators and their eigenstates

$$\hat{N}_\alpha(t) = \hat{A}^\dagger_\alpha(t)\hat{A}_\alpha(t), \quad \hat{N}_\alpha|n_\alpha, t\rangle = n_\alpha|n_\alpha, t\rangle, \tag{19}$$

define the Fock space for each mode. The exact quantum states for each mode are linear superposition of these number states as given by eq. (8), and the wave function for the scalar field is the product of these quantum states.

We now generalize the quadratic form of eq. (10) to a fermionic system by defining the Hamiltonian

$$\begin{aligned} \hat{H}(t) = \sum_\alpha \Omega^{(D)}_\alpha(t)\Big(\hat{b}^\dagger_\alpha\hat{b}_\alpha - \hat{d}_\alpha\hat{d}^\dagger_\alpha\Big) + \Omega^{(--)}_\alpha(t)\hat{b}_\alpha\hat{d}_\alpha + \Omega^{(++)}_\alpha(t)\hat{b}^\dagger_\alpha\hat{d}^\dagger_\alpha \\ + \Omega^{(-+)}_\alpha(t)\hat{b}_\alpha\hat{d}^\dagger_\alpha + \Omega^{(+-)}_\alpha(t)\hat{b}^\dagger_\alpha\hat{d}_\alpha. \end{aligned} \tag{20}$$

Here, α denotes a collective notation for the momentum $\mathbf{k}$ and spinor index r, and $\sum_\alpha = \int d^3\mathbf{k} \sum_{r=1,2}$, and the particle and antiparticle creation and annihilation operators satisfy the anticommutators, respectively,

$$\{\hat{b}_\alpha, \hat{b}^\dagger_\beta\} = \delta_{\alpha\beta}, \quad \{\hat{d}_\alpha, \hat{d}^\dagger_\beta\} = \delta_{\alpha\beta}, \tag{21}$$

and all the other anticommutators vanish. So the Hamiltonian given by eq. (20) describes a system bilinear in the fermion fields. The unitarity of quantum evolution requires the hermiticity of the Hamiltonian which leads to the conditions

$$\Omega^{(D)}_\alpha(t) = \Omega^{(D)*}_\alpha(t), \quad \Omega^{(--)}_\alpha(t) = -\Omega^{(++)*}_\alpha(t), \quad \Omega^{(-+)}_\alpha(t) = -\Omega^{(+-)*}_\alpha(t). \tag{22}$$

It is found that an invariant of the form

$$\hat{\mathcal{O}}_\alpha(t) = f^{(-)}_\alpha(t)\hat{b}_\alpha + f^{(+)}_\alpha(t)\hat{b}^\dagger_\alpha + g^{(-)}_\alpha(t)\hat{d}_\alpha + g^{(+)}_\alpha(t)\hat{d}^\dagger_\alpha, \tag{23}$$

satisfies eq. (2) provided that

$$\begin{aligned} &i\frac{\partial}{\partial t}f^{(-)}_\alpha(t) + \Omega^{(D)}_\alpha(t)f^{(-)}_\alpha(t) - \Omega^{(-+)}_\alpha(t)g^{(-)}_\alpha(t) - \Omega^{(--)}_\alpha(t)g^{(+)}_\alpha(t) = 0, \\ &i\frac{\partial}{\partial t}f^{(+)}_\alpha(t) - \Omega^{(D)}_\alpha(t)f^{(+)}_\alpha(t) - \Omega^{(++)}_\alpha(t)g^{(-)}_\alpha(t) - \Omega^{(+-)}_\alpha(t)g^{(+)}_\alpha(t) = 0, \\ &i\frac{\partial}{\partial t}g^{(-)}_\alpha(t) + \Omega^{(D)}_\alpha(t)g^{(-)}_\alpha(t) + \Omega^{(+-)}_\alpha(t)f^{(-)}_\alpha(t) + \Omega^{(--)}_\alpha(t)f^{(+)}_\alpha(t) = 0, \\ &i\frac{\partial}{\partial t}g^{(+)}_\alpha(t) - \Omega^{(D)}_\alpha(t)g^{(+)}_\alpha(t) + \Omega^{(++)}_\alpha(t)f^{(-)}_\alpha(t) + \Omega^{(-+)}_\alpha(t)f^{(+)}_\alpha(t) = 0. \end{aligned} \tag{24}$$

By introducing two column vectors

$$U_\alpha(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} f_\alpha^{(-)}(t) + f_\alpha^{(+)}(t) \\ f_\alpha^{(-)}(t) - f_\alpha^{(+)}(t) \end{pmatrix}, \quad V_\alpha(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} g_\alpha^{(-)}(t) + g_\alpha^{(+)}(t) \\ g_\alpha^{(-)}(t) - g_\alpha^{(+)}(t) \end{pmatrix}, \tag{25}$$

we can write eq. (24) as two vector equations

$$i\frac{\partial}{\partial t}U_\alpha(t) + \Omega_\alpha^{(D)}(t)\sigma_1 U_\alpha(t) + M_{U_\alpha}(t)V_\alpha(t) = 0, \tag{26}$$

$$i\frac{\partial}{\partial t}V_\alpha(t) + \Omega_\alpha^{(D)}(t)\sigma_1 V_\alpha(t) + M_{V_\alpha}(t)U_\alpha(t) = 0. \tag{27}$$

Here, $M_{U_\alpha}(t)$ and $M_{V_\alpha}(t)$ denote the mixing matrices between $U_\alpha(t)$ and $V_\alpha(t)$:

$$\begin{aligned} M_{U_\alpha}(t) &= \Delta_\alpha^{(0)}(t)I + \Delta_\alpha^{(1)}(t)\sigma_1 + \Delta_\alpha^{(2)}(t)\sigma_2 + \Delta_\alpha^{(3)}(t)\sigma_3, \\ M_{V_\alpha}(t) &= -\Delta_\alpha^{(0)}(t)I + \Delta_\alpha^{(1)}(t)\sigma_1 - \Delta_\alpha^{(2)}(t)\sigma_2 - \Delta_\alpha^{(3)}(t)\sigma_3, \end{aligned} \tag{28}$$

where σ_i are Pauli spin matrices and

$$\begin{aligned} \Delta_\alpha^{(0)}(t) &= -\frac{1}{2}\Big(\Omega_\alpha^{(-+)}(t) - \Omega_\alpha^{(-+)*}(t)\Big), \quad \Delta_\alpha^{(1)}(t) = -\frac{1}{2}\Big(\Omega_\alpha^{(-+)}(t) + \Omega_\alpha^{(-+)*}(t)\Big), \\ \Delta_\alpha^{(2)}(t) &= \frac{i}{2}\Big(\Omega_\alpha^{(--)}(t) + \Omega_\alpha^{(--)*}(t)\Big), \quad \Delta_\alpha^{(3)}(t) = \frac{1}{2}\Big(\Omega_\alpha^{(++)}(t) - \Omega_\alpha^{(++)*}(t)\Big). \end{aligned} \tag{29}$$

By eliminating either $U_\alpha(t)$ or $V_\alpha(t)$ in eq. (27) we obtain the second order differential equations. So we get two types of annihilation operators

$$\begin{aligned} \hat{B}_\alpha(t) &= f_\alpha^{(B-)}(t)\hat{b}_\alpha + f_\alpha^{(B+)}(t)\hat{b}_\alpha^\dagger + g_\alpha^{(B-)}(t)\hat{d}_\alpha + g_\alpha^{(B+)}(t)\hat{d}_\alpha^\dagger, \\ \hat{D}_\alpha(t) &= f_\alpha^{(D-)}(t)\hat{b}_\alpha + f_\alpha^{(D+)}(t)\hat{b}_\alpha^\dagger + g_\alpha^{(D-)}(t)\hat{d}_\alpha + g_\alpha^{(D+)}(t)\hat{d}_\alpha^\dagger, \end{aligned} \tag{30}$$

where $f_\alpha^{(B-)}(t)$ and $g_\alpha^{(D-)}(t)$ are chosen the (adiabatic) positive frequency solutions.

One can make $\hat{B}_\alpha(t), \hat{D}_\alpha(t)$ and their hermitian conjugates the annihilation and creation operators that satisfy the anticommutators at each time

$$\{\hat{B}_\alpha(t), \hat{B}_\beta^\dagger(t)\} = \delta_{\alpha\beta}, \quad \{\hat{D}_\alpha(t), \hat{D}_\beta^\dagger(t)\} = \delta_{\alpha\beta}. \tag{31}$$

These lead to the consistency condition for the coefficient functions

$$\begin{aligned} & f_\alpha^{(B-)*}(t)f_\alpha^{(B-)}(t) + f_\alpha^{(B+)*}(t)f_\alpha^{(B+)}(t) + g_\alpha^{(B-)*}(t)g_\alpha^{(B-)}(t) + g_\alpha^{(B+)*}(t)g_\alpha^{(B+)}(t) = 1, \\ & f_\alpha^{(D-)*}(t)f_\alpha^{(D-)}(t) + f_\alpha^{(D+)*}(t)f_\alpha^{(D+)}(t) + g_\alpha^{(D-)*}(t)g_\alpha^{(D-)}(t) + g_\alpha^{(D+)*}(t)g_\alpha^{(D+)}(t) = 1. \end{aligned} \tag{32}$$

The consistency of anticommutators for all times is guaranteed by

$$\begin{aligned} \frac{\partial}{\partial t}\Big(U^{(B)\dagger}(t)U^{(B)}(t) + V^{(B)\dagger}(t)V^{(B)}(t)\Big) &= 0, \\ \frac{\partial}{\partial t}\Big(U^{(D)\dagger}(t)U^{(D)}(t) + V^{(D)\dagger}(t)V^{(B)}(t)\Big) &= 0, \end{aligned} \tag{33}$$

where $M_{U_\alpha}^\dagger(t) = M_{V_\alpha}(t)$ is used. The other anticommutators

$$\begin{aligned} \{\hat{B}_\alpha(t), \hat{B}_\beta(t)\} &= \{\hat{B}_\alpha^\dagger(t), \hat{B}_\beta^\dagger(t)\} = 0, \\ \{\hat{D}_\alpha(t), \hat{D}_\beta(t)\} &= \{\hat{D}_\alpha^\dagger(t), \hat{D}_\beta^\dagger(t)\} = 0, \end{aligned} \tag{34}$$

are consistently satisfied for all times due to

$$\frac{\partial}{\partial t}\Big(U^{(B)\dagger}(t)\sigma_3 U^{(B)}(t) + V^{(B)\dagger}(t)\sigma_3 V^{(B)}(t)\Big) = 0,$$
$$\frac{\partial}{\partial t}\Big(U^{(D)\dagger}(t)\sigma_3 U^{(D)}(t) + V^{(D)\dagger}(t)\sigma_3 V^{(B)}(t)\Big) = 0, \tag{35}$$

where $\sigma_3 M^T_{U_\alpha}(t)\sigma_3 = -M_{V_\alpha}(t)$ is used. Still other anticommutators

$$\{\hat{B}_\alpha(t), \hat{D}_\beta(t)\} = \{\hat{B}^\dagger_\alpha(t), \hat{D}^\dagger_\beta(t)\} = 0,$$
$$\{\hat{B}_\alpha(t), \hat{D}^\dagger_\beta(t)\} = \{\hat{B}^\dagger_\alpha(t), \hat{D}_\beta(t)\} = 0, \tag{36}$$

are also satisfied for all times due to

$$\frac{\partial}{\partial t}\Big(U^{(B)T}(t)\sigma_3 U^{(D)}(t) + V^{(B)T}(t)\sigma_3 V^{(D)}(t)\Big) = 0,$$
$$\frac{\partial}{\partial t}\Big(U^{(B)\dagger}(t) U^{(D)}(t) + V^{(B)\dagger} V^{(D)}_\alpha\Big) = 0. \tag{37}$$

Since the $\hat{B}_\alpha(t), \hat{B}^\dagger_\alpha(t)$ and $\hat{D}_\alpha(t), \hat{D}^\dagger_\alpha(t)$ share the same properties of the annihilation and creation operators for time-independent system, they will be used as those for time-dependent system. So the number operators are defined by

$$\hat{N}^{(B)}_\alpha(t) = \hat{B}^\dagger_\alpha(t)\hat{B}_\alpha(t), \quad \hat{N}^{(D)}_\alpha(t) = \hat{D}^\dagger_\alpha(t)\hat{D}_\alpha(t). \tag{38}$$

The eigenstates for each mode are the number states of $\hat{N}^{(B)}_\alpha(t)$ or $\hat{N}^{(D)}_\alpha(t)$. For instance, the zero-particle state for $\hat{N}^{(B)}_\alpha(t)$ is given by

$$|0^{(B)}_\alpha, t\rangle = c^{(B)}_{\alpha 0}(t)|0^{(b)}_\alpha, 0^{(d)}_\alpha\rangle + c^{(B)}_{\alpha 1}(t)|0^{(b)}_\alpha, 1^{(d)}_\alpha\rangle + e^{(B)}_{\alpha 0}(t)|1^{(b)}_\alpha, 0^{(d)}_\alpha\rangle + e^{(B)}_{\alpha 1}(t)|1^{(b)}_\alpha, 1^{(d)}_\alpha\rangle, \tag{39}$$

where $|n'^{(b)}, n^{(d)}\rangle$ are the number states with respect to $\hat{b}^\dagger_\alpha \hat{b}_\alpha$ and $\hat{d}^\dagger_\alpha \hat{d}_\alpha$, respectively, and

$$f^{(B-)}_\alpha(t) f^{(B+)}_\alpha(t) - g^{(B-)}_\alpha(t) g^{(B+)}_\alpha(t) = 0. \tag{40}$$

One-particle state is constructed by applying the operator $\hat{B}^\dagger_\alpha(t)$ to the zero-particle state (eq. (39)). The number states of the antiparticle can be similarly constructed by using the number operator $\hat{N}^{(D)}_\alpha(t)$. These states in general depend on time, but their eigenvalues (occupation numbers) do not depend on time as expected according to the Lewis-Riesenfeld theorem. The vacuum state is the one that is annihilated by all the annihilation operators $\hat{B}_\alpha(t)$ and $\hat{D}_\alpha(t)$:

$$\hat{B}_\alpha(t)|0, 0, t\rangle = \hat{D}_\alpha(t)|0, 0, t\rangle = 0. \tag{41}$$

When the interaction is turned on for a finite period of time or the system evolves from an asymptotic region where all the Ω_α are constants and eq. (41) coincides with the Minkowski vacuum, the pair production of fermions from the initial vacuum is given by

$$\langle 0, 0| \sum_\alpha \Big(\hat{N}^{(B)}_\alpha(t) + \hat{N}^{(D)}_\alpha(t)\Big)|0, 0\rangle = \sum_\alpha \Big(f^{(B+)*}_\alpha(t) f^{(B+)}_\alpha + g^{(D+)*}_\alpha(t) g^{(D+)}_\alpha\Big). \tag{42}$$

Let us consider some particular examples of using the Hamiltonian given by eq. (20).
Dirac Oscillator. It is the case where $\Omega^{(D)}_\alpha \neq 0$ and $\Omega^{(\pm,\pm)}_\alpha = 0$. Then the first order equations (eq. (26)) can be easily integrated as $\hat{B}_\alpha(t) = e^{i\int \Omega^{(D)}_\alpha} \hat{b}_\alpha$ and $\hat{D}_\alpha(t) = e^{i\int \Omega^{(D)}_\alpha} \hat{d}_\alpha$ and their hermitian conjugates. So the generalized invariants become $\hat{B}^\dagger_\alpha(t)\hat{B}_\alpha(t) = \hat{b}^\dagger_\alpha \hat{b}_\alpha$ and $\hat{D}^\dagger_\alpha(t)\hat{D}_\alpha(t) = \hat{d}^\dagger_\alpha \hat{d}_\alpha$. Their number states are not only the eigenstates of the generalized invariants but also those that diagonalize the Hamiltonian.
Fermion Scalar Field Coupling. The case where $\Omega^{(-+)}_\alpha = \Omega^{(+-)}_\alpha = 0$, $\Omega^{(D)}_\alpha = k_0 + \frac{mg\phi(t)}{2k_0}$ and $\Omega^{(--)}_\alpha = -\Omega^{(++)}_\alpha = \frac{mg\phi(t)}{2k_0}$ leads to the Hamiltonian

$$\hat{H}(t) = \int d^3\mathbf{k}\Big[\Big(k_0 + \frac{mg\phi(t)}{2k_0}\Big)\sum_{r=1,2}\Big\{\hat{b}^\dagger_r(\mathbf{k})\hat{b}_r(\mathbf{k}) - \hat{d}_r(\mathbf{k})\hat{d}^\dagger_r(\mathbf{k})\Big\} + \frac{mg\phi(t)}{2k_0}\sum_{r=1,2}\Big\{\hat{b}^\dagger_r(\mathbf{k})\hat{d}^\dagger_r(\mathbf{k}) + \hat{d}_r(\mathbf{k})\hat{b}_r(\mathbf{k})\Big\}\Big]. \tag{43}$$

This describes a fermion coupled to a uniform scalar field through the Yukawa coupling

$$\mathcal{L} = \frac{1}{2}\Big(\overline{\psi}i\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})i\gamma^\mu\psi\Big) - m\overline{\psi}\psi - \lambda\phi(t)\overline{\psi}\psi, \tag{44}$$

where λ is a coupling constant. The coefficient functions are found to satisfy

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\Big(f^{(-)}_\alpha + g^{(+)}\Big) - \frac{1}{\Omega^{(D)}_\alpha}\frac{\partial\Omega^{(D)}_\alpha}{\partial t}\Big(f^{(-)}_\alpha + g^{(+)}\Big) \\
&\quad + \Big\{\Omega^{(D)2}_\alpha + \Omega^{(--)2}_\alpha + i\Omega^{(D)}_\alpha\frac{\partial}{\partial t}\Big(\frac{\Omega^{(--)}_\alpha}{\Omega^{(D)}_\alpha}\Big)\Big\}\Big(f^{(-)}_\alpha + g^{(+)}\Big) = 0, \\
&\frac{\partial^2}{\partial t^2}\Big(f^{(-)}_\alpha - g^{(+)}\Big) - \frac{1}{\Omega^{(D)}_\alpha}\frac{\partial\Omega^{(D)}_\alpha}{\partial t}\Big(f^{(-)}_\alpha - g^{(+)}\Big) \\
&\quad + \Big\{\Omega^{(D)2}_\alpha + \Omega^{(--)2}_\alpha - i\Omega^{(D)}_\alpha\frac{\partial}{\partial t}\Big(\frac{\Omega^{(--)}_\alpha}{\Omega^{(D)}_\alpha}\Big)\Big\}\Big(f^{(-)}_\alpha - g^{(+)}\Big) = 0.
\end{aligned} \tag{45}$$

The positive solution $f^{(-)}_\alpha(t)$ and the negative one $g^{(+)}(t)$ with the condition (32) give the correct pair production rate. Interestingly enough the free fermion in the Friedmann-Robertson-Walker universe has the same form of Hamiltonian as eq. (43) [6].

A few comments are in order. The generalized invariants in eq. (30) for the Hamiltonian given by eq. (20) are more general than those for eq. (43), since $\hat{B}_\alpha(t)$ involves only $\hat{b}_\alpha$ and $\hat{d}^\dagger_\alpha$ and $\hat{D}_\alpha(t)$ involve only $\hat{b}^\dagger_\alpha$ and $\hat{d}_\alpha$. When one has cross terms like $\hat{b}_\alpha\hat{d}^\dagger_\alpha$ and $\hat{b}^\dagger_\alpha\hat{d}_\alpha$, then eq. (30) is the unique choice compatible with the algebra. Furthermore, eq. (42), when restricted to eq. (43) gives the same result for the pair production as other quantum field theories. One may also extend the analysis to the fermion interacting with an electromagnetic field by introducing a Hamiltonian that is similar in form to eq. (20) but involves Dirac matrices γ_μ in a coupling with the time dependent coefficient $\Omega_\alpha(t)$.

One of the important issues not treated in this paper is the non-equilibrium aspect of open system of quantum fields. This can be done by introducing a density operator of the form $\hat{\rho}_\alpha(t) = e^{-\beta(\hat{N}^{(B)}_\alpha(t) + \hat{N}^{(D)}_\alpha(t))}$, which satisfies the Liouville-von Neumann equation by its construction, which will be addressed in a future publication. One of the applications of the result of this paper is the evolution of fermions in an expanding Universe, where fermions gain time-dependence through the interaction with the gravity. The other is to calculate the pair production rate of fermions due to the oscillating inflaton and to provide a preheating mechanism for the fermionic system. Finally the time evolution of the quark-gluon plasma in a relativistic heavy ion collision would simulate the conditions present at the early moments of the Univese and would thus provide us with an experimental view of the evolution of the Universe.

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